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Group Rings Satisfying a Polynomial Identity

D. S. PASSMAN

Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706

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1. INTRODUCTION

Let K be a field and let G be a (not necessarily finite) group. We let $K[G]$ denote the group ring of G over K. That is, $K[G]$ is a K-algebra with basis $\{x \mid x \in G\}$ and with multiplication defined distributively using the group multiplication in G.

Let $K[\zeta_1, \zeta_2, ...]$ be the polynomial ring over K in the noncommuting indeterminates $\zeta_1, \zeta_2, ...$. An algebra E over K is said to satisfy a polynomial identity if there exists $f(\zeta_1, \zeta_2, ..., \zeta_n) \in K[\zeta_1, \zeta_2, ...]$, $f \neq 0$ with

$$f(\alpha_1, \alpha_2, ..., \alpha_n) = 0$$

for all $\alpha_1, \alpha_2, ..., \alpha_n \in E$. For example, any commutative algebra satisfies $f(\zeta_1, \zeta_2) = \zeta_1\zeta_2 - \zeta_2\zeta_1$.

In 1949, I. Kaplansky [3] initiated the algebraic study of group rings by showing that if G has an Abelian subgroup A of finite index n, then the group ring $K[G]$ satisfies a polynomial identity of degree $n^2 + 1$. Twelve years later in 1961, S. A. Amitsur [1] sharpened the above result to get a polynomial identity of degree $2n$ and then he proved a partial converse. He showed that if K is a field of characteristic 0 and if $K[G]$ satisfies a polynomial identity of degree $\leqslant 4$ then G has an Abelian subgroup A with $[G : A] \leqslant 4$. The characteristic 0 case was completed by I. M. Isaacs and this author [2] in 1964. We showed that if $K[G]$ satisfies a polynomial identity of degree n, then G has an Abelian subgroup A with $[G : A] \leqslant J(n)$. Here J is the function associated with Jordan's theorem on finite complex linear groups.

The first results in characteristic $p > 0$ were obtained by Martha Smith in her recent thesis (University of Chicago, 1970, [7]). She showed, for example, that if $K[G]$ is a prime ring and satisfies a polynomial identity of degree n, then G has an Abelian subgroup A with $[G : A] \leqslant (n/2)^2$ and she also made the following crucial observation. Let $\Delta(G)$ denote the $F.C.$ subgroup of G, that is, $\Delta(G)$ is the set of all elements of G having only finitely many conjugates. If A is an Abelian subgroup of finite index, then clearly $A \subseteq \Delta(G)$ and hence $\Delta(G)$ has finite index. Thus she observed that a first step in finding such an Abelian subgroup A is to show that $[G : \Delta(G)] < \infty$.

Motivated by [7], this author then obtained the following results in [5]. If $K[G]$ satisfies a polynomial identity of degree n, then $[G : \Delta(G)] \leqslant n!$. If, in addition, $K[G]$ is a semiprime ring, then G has an Abelian subgroup A with $[G : A] \leqslant n!\, J(n)$. Finally, an example was offered, namely an infinite extra-special p-group in characteristic p, to show that if $K[G]$ is not semiprime, then G need not have an Abelian subgroup of finite index.

In this paper we completely solve the problem. We offer necessary and sufficient conditions for the group ring $K[G]$ to satisfy a polynomial identity. The results are as follows.

THEOREM 1.1 [2]. *Let K be a field of characteristic 0 and let A denote an Abelian subgroup of G.*

(i) *If $[G : A] < \infty$, then $K[G]$ satisfies a polynomial identity of degree $2[G : A]$.*

(ii) *If $K[G]$ satisfies a polynomial identity of degree n, then G has such a subgroup A with $[G : A]$ bounded by a fixed function of n.*

COROLLARY 1.2. *Let K be a field of characteristic 0. Then $K[G]$ satisfies a polynomial identity if and only if G has an Abelian subgroup of finite index.*

Let p be a prime. We say that A is a p-Abelian group if A', the commutator subgroup of A, is a finite p-group.

THEOREM 1.3. *Let K be a field of characteristic $p > 0$ and let A denote a p-Abelian subgroup of G.*

(i) *If $[G : A] < \infty$, then $K[G]$ satisfies a polynomial identity of degree $2[G : A] \cdot | A' |$.*

(ii) *If $K[G]$ satisfies a polynomial identity of degree n, then G has such a subgroup A with $[G : A] \cdot | A' |$ bounded by a fixed function of n.*

COROLLARY 1.4. *Let K be a field of characteristic $p > 0$. Then $K[G]$ satisfies a polynomial identity if and only if G has a p-Abelian subgroup of finite index.*

The corollaries of course follow immediately from the theorems. We now prove the easy direction of these theorems.

Proof of Theorem 1.1 (i) *and Theorem* 1.3 (i). Let A be the appropriate subgroup of G. Suppose that $[G : A] = n$ and let $x_1, x_2, \ldots, x_n$ be a set of right coset representatives of A in G. Let $E = K[A]$ and $V = K[G]$. Then, clearly, V is a left E-module with basis $\{x_1, x_2, \ldots, x_n\}$. Now V is also a right $K[G]$-module and as such it is faithful. Since right and left multiplication commute as operators on V, it follows that $K[G]$ is a set of E-linear transformations on an n-dimensional free E-module V. Thus $K[G] \subseteq E_n$, the ring of $n \times n$ matrices over E.

Suppose first that A is Abelian. Then E is a commutative algebra and hence by the Amitsur–Levitzki theorem [6, Theorem 4.6] E_n satisfies $s_{2n}(\zeta_1, \zeta_2, \ldots, \zeta_{2n})$ the standard polynomial of degree $2n$. This yields Theorem 1.1 (i). Now let K have characteristic $p > 0$ and let A be p-Abelian. Let I denote the kernel of the natural epimorphism $K[A'] \to K[A'/A'] = K$. Since A' is a finite p-group it follows that I is the radical of $K[A']$; so $I^m = 0$ where $m = |A'|$. Since A' is normal in A, it follows also that $I \cdot K[A] = K[A] \cdot I$ so that $I \cdot E$ is an ideal of E with $(I \cdot E)^m = 0$. Moreover, $E/(I \cdot E) \simeq K[A/A']$ is commutative. Consider the natural epimorphism $E_n \to (E/(I \cdot E))_n$. Since $E/(I \cdot E)$ is commutative and since the kernel of this map, $(I \cdot E)_n$, is nilpotent of exponent $\leqslant m$, we conclude that E_n satisfies $s_{2n}(\zeta_1, \zeta_2, \ldots, \zeta_{2n})^m$ a polynomial of degree $2nm$. This yields Theorem 1.3 (i).

2. Subsets of Finite Index

Let G be a group and let T be a subset of G. We say that T has finite index in G if there exists $x_1, x_2, \ldots, x_k \in G$ for some finite k with

$$G = Tx_1 \cup Tx_2 \cup \cdots \cup Tx_k .$$

We then define the index $[G : T]$ to be the minimum possible such integer k. Observe that if T is a subgroup of G, then this agrees with the usual definition of index.

Lemma 2.1 *Let $S = \bigcup_1^k H_i g_i$ be a finite union of cosets of the subgroups H_i of G and assume that $S \neq G$. Then there exists $x_1, x_2, \ldots, x_t \in G$ with $t = (k + 1)!$ such that $\bigcap_1^t Sx_i = \phi$. Thus if T is a subset of G with $G = S \cup T$, then $[G : T] \leqslant (k + 1)!$.*

Proof. We prove the existence of the x_i by induction on k. Let $k = 1$. Then $H_1 g_1 \neq G$; so there exists another coset of H_1 say $H_1 x_1$. Then

$$S \cap S g_1^{-1} x_1 = H_1 g_1 \cap H_1 x_1 = \phi$$

and this case follows.

Assume the result for $k-1$ and set $r = k!$. Fix a subscript i. Since $S \neq G$, S cannot contain all the right cosets of H_i; so say $S \not\supseteq H_i x_i$. Thus $Sx_i^{-1} \not\supseteq H_i$ and

$$H_i > H_i \cap Sx_i^{-1} = \bigcup_{j=1}^{k} H_i \cap H_j g_j x_i^{-1}.$$

Now either $H_i \cap H_j g_j x_i^{-1}$ is empty or $H_i \cap H_j g_j x_i^{-1} = (H_i \cap H_j) z_{ij}$ for some $z_{ij} \in H_i$. Thus considering the nonempty intersections yields

$$H_i > \bigcup{}' (H_i \cap H_j) z_{ij}$$

and observe that $j = i$ is missing here since $S \supseteq H_i g_i$, $S \not\supseteq H_i x_i$ yields $H_i \cap H_i g_i x_i^{-1} = \phi$.

Thus by induction there exists $y_{i1}, y_{i2}, \ldots, y_{ir} \in H_i$ with

$$H_i \cap \bigcap_{j=1}^{r} Sx_i^{-1} y_{ij} = \bigcap_{j=1}^{r} (H_i \cap Sx_i^{-1}) y_{ij} = \phi$$

and therefore

$$H_i g_i \cap \bigcap_{j=1}^{r} Sx_i^{-1} y_{ij} g_i = \phi.$$

Since $S = \bigcup_1^k H_i g_i$, this yields

$$S \cap \bigcap_{i=1}^{k} \bigcap_{j=1}^{r} Sx_i^{-1} y_{ij} g_i = \phi.$$

Now the number of right translates of S in the above is

$$kr + 1 \leqslant (k+1)r = (k+1)!\,;$$

so the first result follows by induction.

Now suppose $\bigcap_1^t Sx_i = \phi$ and $G = S \cup T$. Since $G = Sx_i \cup Tx_i$, this yields clearly $G = \bigcup_1^t Tx_i$ and thus $[G : T] \leqslant t = (k+1)!$. The result follows.

We now offer some conditions that guarantee that unions of the above type S are not equal to G. The following is Lemma 5.2 of [6].

LEMMA 2.2. *Let G be a group and suppose that G can be written as $G = \bigcup H_i g_i$, a finite union of cosets of subgroups H_i of G. Then $G = \bigcup' H_i g_i$ where the union is restricted to those H_i with $[G : H_i] < \infty$.*

LEMMA 2.3. *Let $S = \bigcup_1^k H_i g_i$ be a finite union of cosets of subgroups H_i of G. If $[G : H_i] > k$ for all i, then $S \neq G$.*

Proof. Suppose by way of contradiction that $S = G$. Then by Lemma 2.2 we may assume that $[G : H_i]$ is finite for all i. Then there exists a normal subgroup W of G of finite index such that $H_i \supseteq W$ for all i. Let $^-$ denote the natural map $G \to G/W$. Then $G = \bigcup H_i g_i$ yields $\bar{G} = \bigcup \bar{H}_i \bar{g}_i$. Thus

$$|\bar{G}| \leqslant \sum_1^k |\bar{H}_i| \leqslant k\,|\bar{H}_j|$$

for some j and hence $[G : H_j] = [\bar{G} : \bar{H}_j] \leqslant k$, a contradiction.

If T is a subset of G, let $T^* = T \cup \{1\} \cup T^{-1}$.

LEMMA 2.4. *Let T be a subset of G with $[G : T] \leqslant k$. Then*

$$(T^*)^{4^k} = T^* \cdot T^* \cdot \cdots \cdot T^* \qquad (4^k \text{ times})$$

is a subgroup of G.

Proof. By induction on k, the case $k = 1$ being clear. Since $T^* \supseteq T$ we have $[G : T^*] \leqslant k$ and thus we may assume that $T = T^*$. Write

$$G = Tx_1 \cup Tx_2 \cup \cdots \cup Tx_k$$

and multiplying on the right by x_1^{-1} if necessary we can suppose that $x_1 = 1$. If T is a subgroup of G, the result is trivial; so suppose it is not. Then $T = T^*$ implies that $T^2 \not\subseteq T$ and hence say $T^2 \cap Tx_k \neq \phi$. It then follows that $x_k \in T^3$ and $Tx_k \subseteq T^4$. Therefore, $T^4 \supseteq Tx_1 \cup Tx_k$; so

$$G = T^4x_1 \cup T^4x_2 \cup \cdots \cup T^4x_{k-1}$$

and $[G : T^4] \leqslant k - 1$. Since clearly $T^4 = (T^4)^*$, we have by induction

$$T^{4^k} = (T^4)^{4^{k-1}}$$

is a subgroup of G.

3. POLYNOMIAL IDENTITIES

Let G be a group. For each integer k we define

$$\Delta_k = \Delta_k(G) = \{x \in G \mid [G : \mathbb{C}(x)] \leqslant k\}.$$

Thus Δ_k is a normal subset of G and $\Delta_k = \Delta_k{}^*$ in the notation of Lemma 2.4. However, Δ_k need not be a subgroup of G.

Let K be a field. Then we define $\theta_k : K[G] \to K[G]$ by

$$\alpha = \sum_{x \in G} a_x x \to \theta_k(\alpha) = \sum_{x \in \Delta_k} a_x .$$

This is merely a K-linear map of $K[G]$ into $K[G]$. It is not in general a ring homomorphism.

If $\alpha = \sum a_x x \in K[G]$ we define the support of α to be

$$\operatorname{supp} \alpha = \{x \in G \mid a_x \neq 0\}.$$

Then $\operatorname{supp} \alpha$ is a finite subset of G and $\theta_k(\alpha) = 0$ if and only if $(\operatorname{supp} \alpha) \cap \Delta_k = \phi$.

LEMMA 3.1. *Let* $\alpha_1, \alpha_2, \ldots, \alpha_t, \beta_1, \beta_2, \ldots, \beta_t, \gamma \in K[G]$ *with* $\theta_k(\alpha_i) = 0$ *for all* i. *Suppose that*

$$\left| \bigcup_i \operatorname{supp} \alpha_i \right| = r, \qquad \left| \bigcup_i \operatorname{supp} \beta_i \right| = s$$

and $rs < k$. *Let* T *be a subset of* G *and suppose that for all* $x \in G - T$ *we have*

$$\alpha_1 x \beta_1 + \alpha_2 x \beta_2 + \cdots + \alpha_t x \beta_t = x\gamma.$$

Then either $\gamma = 0$ *or* $[G : T] \leq k!$.

Proof. Let

$$\bigcup_i \operatorname{supp} \alpha_i = \{y_1, y_2, \ldots, y_r\},$$

$$\bigcup_i \operatorname{supp} \beta_i = \{z_1, z_2, \ldots, z_s\}.$$

We assume that $\gamma \neq 0$ and let $v \in \operatorname{supp} \gamma$. If y_i is conjugate to vz_j^{-1} in G for some i, j choose $h_{ij} \in G$ with $h_{ij}^{-1} y_i h_{ij} = vz_j^{-1}$.

Let $x \in G - T$. Then by hypothesis we have

$$x^{-1}\alpha_1 x\beta_1 + x^{-1}\alpha_2 x\beta_2 + \cdots + x^{-1}\alpha_t x\beta_t = \gamma.$$

Since $v \in \operatorname{supp} \gamma$ it follows that for some i, j we have $x^{-1} y_i x z_j = v$. Thus

$$x^{-1} y_i x = vz_j^{-1} = h_{ij}^{-1} y_i h_{ij}$$

and $x \in \mathbb{C}(y_i) h_{ij}$.

We have therefore shown that $G = T \cup S$ where

$$S = \bigcup_{ij} \mathbb{C}(y_i)\, h_{ij} .$$

Since $\theta_k(\alpha_i) = 0$ we have $[G : \mathbb{C}(y_i)] > k$ for all i and since there are at most $rs < k$ cosets in the above union we conclude from Lemma 2.3 that $S \neq G$. Thus, by Lemma 2.1,

$$[G : T] \leqslant (rs + 1)! \leqslant k!$$

and the result follows.

The following linearization is due to Kaplansky [6, Lemma 4.1]. Let S_n denote the symmetric group of degree n.

LEMMA 3.2. *Suppose E is an algebra over K which satisfies a nontrivial polynomial identity of degree n. Then E satisfies the polynomial identity $f \in K[\zeta_1, \zeta_2, \ldots, \zeta_n]$ with*

$$f = \sum_{\sigma \in S_n} a_\sigma \zeta_{\sigma(1)} \zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)},$$

where $a_\sigma \in K$ and they are not all zero.

A linear monomial is an element $\mu \in K[\zeta_1, \zeta_2, \ldots, \zeta_n]$ of the form $\mu = \zeta_{i_1} \zeta_{i_2} \cdots \zeta_{i_r}$ with all i_j distinct and with $r \geqslant 1$. Thus μ is linear in each variable.

LEMMA 3.3. *The number of linear monomials in $K[\zeta_1, \zeta_2, \ldots, \zeta_n]$ is $\leqslant (n + 1)!$.*

Proof. The number of linear monomials in $K[\zeta_1, \zeta_2, \ldots, \zeta_n]$ of degree n is of course $n!$. Now any other linear monomial is clearly just an initial segment of one of these. This yields the bound $n \cdot n! \leqslant (n + 1)!$.

THEOREM 3.4. *Suppose $K[G]$ satisfies a polynomial identity of degree n. If $k = (n!)^2$ then*

$$[G : \Delta_k(G)] < (k + 1)!.$$

Proof. We assume by way of contradiction that $[G : \Delta_k(G)] \geqslant (k + 1)!$. By Lemma 3.2 we may assume that $K[G]$ satisfies the polynomial identity

$$f(\zeta_1, \zeta_2, \ldots, \zeta_n) = \zeta_1 \zeta_2 \cdots \zeta_n + \sum_{\substack{\sigma \in S_n \\ \sigma \neq 1}} a_\sigma \zeta_{\sigma(1)} \zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)},$$

so that clearly $n > 1$. For $j = 1, 2, \ldots, n$ define $f_j \in K[\zeta_j, \zeta_{j+1}, \ldots, \zeta_n]$ by

$$f = \zeta_1 \zeta_2 \cdots \zeta_{j-1} f_j + \text{terms not starting with } \zeta_1 \zeta_2 \cdots \zeta_{j-1}.$$

Then clearly $f_1 = f$, $f_n = \zeta_n$ and f_j is a homogeneous multilinear polynomial of degree $n - j + 1$. In particular for all j, ζ_j occurs in each monomial of f_j. We clearly have

$$f_j = \zeta_j f_{j+1} + \text{terms not starting with } \zeta_j.$$

For each $j = 2, 3, \ldots, n$ let $\mathcal{M}_j$ denote the set of all linear monomials in $K[\zeta_j, \zeta_{j+1}, \ldots, \zeta_n]$ and let $\mathcal{M}_1$ be empty. Then for all j, $|\mathcal{M}_j| \leqslant |\mathcal{M}_2| \leqslant n!$ by Lemma 3.3. We show now by induction on $j = 1, 2, \ldots, n$ that for any $x_j, x_{j+1}, \ldots, x_n \in G$ then either $f_j(x_j, x_{j+1}, \ldots, x_n) = 0$ or $\mu(x_j, x_{j+1}, \ldots, x_n) \in \Delta_k$ for some $\mu \in \mathcal{M}_j$. Since $f = f_1$ is a polynomial identity satisfied by $K[G]$, the result for $j = 1$ is clear.

Suppose the result holds for some $j < n$. Fix $x_{j+1}, x_{j+2}, \ldots, x_n \in G$ and let $x \in G$ play the role of the jth variable. Let $\mu \in \mathcal{M}_{j+1}$. If $\mu(x_{j+1}, x_{j+2}, \ldots, x_n) \in \Delta_k$ we are done. Thus we may assume that $\mu(x_{j+1}, x_{j+2}, \ldots, x_n) \notin \Delta_k$ for all $\mu \in \mathcal{M}_{j+1}$. Set $\mathcal{M}_j - \mathcal{M}_{j+1} = \mathcal{T}_j$.

Now let $\mu \in \mathcal{T}_j$ so that μ involves the variable ζ_j. Write $\mu = \mu'\zeta_j\mu''$ where μ' and μ'' are monomials in $K[\zeta_{j+1}, \zeta_{j+2}, \ldots, \zeta_n]$. Then $\mu(x, x_{j+1}, \ldots, x_n) \in \Delta_k$ if and only if

$$x \in \mu'(x_{j+1}, \ldots, x_n)^{-1} \Delta_k \mu''(x_{j+1}, \ldots, x_n)^{-1} = \Delta_k h_\mu$$

a fixed right translate of Δ_k since Δ_k is a normal subset of G. Thus it follows that for all $x \notin T = \bigcup_{\mu \in \mathcal{T}_j} \Delta_k h_\mu$ we have $\mu(x, x_{j+1}, \ldots, x_n) \notin \Delta_k$ for all $\mu \in \mathcal{M}_j$ since $\mathcal{M}_j \subseteq \mathcal{M}_{j+1} \cup \mathcal{T}_j$. Since the inductive result holds for j we conclude that for all $x \notin T$ we have $f_j(x, x_{j+1}, \ldots, x_n) = 0$.

Write

$$f_j(\zeta_j, \zeta_{j+1}, \ldots, \zeta_n) = \zeta_j f_{j+1} - \sum_{i=1}^{t} \alpha_i \zeta_j \beta_i$$

where $\alpha_i, \beta_i \in K[\zeta_{j+1}\zeta_{j+2}, \ldots, \zeta_n]$ and α_i is a linear monomial. Hence $\alpha_i \in \mathcal{M}_{j+1}$. Now by the above we have

$$x f_{j+1}(x_{j+1}, \ldots, x_n) = \sum_i \alpha_i(x_{j+1}, \ldots, x_n)\, x \beta_i(x_{j+1}, \ldots, x_n)$$

for all $x \in G - T$.

We apply Lemma 3.1 with $\gamma = f_{j+1}(x_{j+1}, \ldots, x_n)$. Now f has at most $n!$ monomials and thus clearly in the notation of that lemma, $r \leqslant n! - 1$, $s \leqslant n!$. Hence

$$rs \leqslant (n! - 1)\, n! < (n!)^2 = k.$$

Also $\theta_k(\alpha_i(x_{j+1}, \ldots, x_n)) = 0$ since $\alpha_i \in \mathcal{M}_{j+1}$ implies that

$$\alpha_i(x_{j+1}, \ldots, x_n) \in G - \Delta_k .$$

Thus the hypotheses of Lemma 3.1 are satisfied and there are two possible conclusions.

If $[G : T] \leqslant k!$, then, since $T = \bigcup_{\mu \in \mathscr{T}_j} \Delta_k h_\mu$ and $|\mathscr{T}_j| \leqslant |\mathscr{M}_j| \leqslant n!$, we see that

$$[G : \Delta_k] \leqslant |\mathscr{T}_j| \, k! < (k+1)!,$$

a contradiction by assumption. Thus we conclude that

$$0 = \gamma = f_{j+1}(x_{j+1}, x_{j+2}, \ldots, x_n)$$

and the induction step is proved.

In particular, the inductive result holds for $j = n$. Here $f_n(\zeta_n) = \zeta_n$ and $\mathscr{M}_n = \{\zeta_n\}$. Thus we conclude that for all $x \in G$ we have either $x = 0$ or $x \in \Delta_k$, a contradiction since $G \neq \Delta_k$. Therefore the assumption that $[G : \Delta_k] \geqslant (k+1)!$ is false and the theorem is proved.

COROLLARY 3.5. *Suppose $K[G]$ satisfies a polynomial identity of degree n and set $k = (n!)^2$. Then G has a characteristic subgroup G_0 such that $[G : G_0] < (k+1)!$ and such that for all $x \in G_0$*

$$[G: \mathbb{C}_G(x)] \leqslant k^{4^{(k+1)!}}.$$

Proof. Let $G_0 = \langle \Delta_k(G) \rangle$ so that G_0 is certainly a characteristic subgroup of G. By Theorem 3.4, $[G : \Delta_k(G)] < (k+1)!$ so clearly $[G : G_0] < (k+1)!$. Since $\Delta_k = \Delta_k{}^*$, Lemma 2.4 implies that $(\Delta_k)^{4^{(k+1)!}}$ is a subgroup of G which is therefore equal to G_0. Thus every element of G_0 is a product of $4^{(k+1)!}$ elements of Δ_k. Since each element of Δ_k has at most k conjugates in G, it now follows easily that each element of G_0 has at most $k^{4^{(k+1)!}}$ conjugates in G. This completes the proof.

4. GROUPS WITH SMALL CLASSES

The group G_0 of the previous result has the property that all its conjugacy classes are of finite bounded size. In this section we study such groups in general. The following is Dietzmann's lemma [4, p. 154].

LEMMA 4.1. *Let $S = \{x_1, x_2, \ldots, x_s\}$ be a finite normal subset of G. Then every element in $\langle S \rangle$, the subgroup generated by S, can be written as*

$$x_1^{a_1} x_2^{a_2} \cdots x_s^{a_s}$$

for suitable integers a_i. In particular, if each x_i has finite order $\leqslant r$ then $|\langle S \rangle| \leqslant r^s$.

LEMMA 4.2. *Let k be an integer and suppose that $G = \Delta_k(G)$. Let $x \in G$ and let G_x denote the normal closure of (x, G). Then*

$$|G_x| \leq (k^4)^{k^2}.$$

Proof. We first observe that every commutator in G has order $\leq k^4$ as follows. Consider the commutator (x, y) and set $H = \langle x, y \rangle$. Then $Z = H \cap \mathbb{C}(x) \cap \mathbb{C}(y)$ is central in H and $[H : Z] \leq k^2$. Thus $(x, y)^j \in Z$ for some $j \leq k^2$. By considering the transfer homomorphism $\tau : H \to Z$ we conclude easily that every element of $Z \cap H'$ has order dividing $[H : Z]$ and thus this fact follows.

Now G_x is generated by the $\leq k$ commutators of x and their $\leq k$ conjugates. Thus G_x has $\leq k^2$ generators forming a normal subset of G. Moreover, each such generator has period $\leq k^4$ by the above. Thus by Lemma 4.1, $|G_x| \leq (k^4)^{k^2}$.

LEMMA 4.3. *Let $G = \Delta_k(G)$ and let $x \in G - \mathbb{Z}(G)$. Suppose that W is a normal subgroup of G with $(x, G) \subseteq W$. If $H = \mathbb{C}(x)/W \cap \mathbb{C}(x)$, then $H = \Delta_{k-1}(H)$.*

Proof. Let y^G denote the conjugacy class of y in G. Now certainly $H = \Delta_k(H)$ and we suppose by way of contradiction that $H \neq \Delta_{k-1}(H)$. Thus there exists $\tilde{y} \in H$ with $|\tilde{y}^H| = k$. Let y be an inverse image for $\tilde{y}$ in $\mathbb{C}(x)$. (Note here all centralizers are in G.) Since $y^{\mathbb{C}(x)}$ maps onto $\tilde{y}^H$ we have

$$k \geq |y^G| \geq |y^{\mathbb{C}(x)}| \geq |\tilde{y}^H| = k,$$

and thus we conclude that $y^G = y^{\mathbb{C}(x)}$, $|y^{\mathbb{C}(x)}| = k$, and that all elements of $y^G = y^{\mathbb{C}(x)}$ are in distinct cosets of $\mathbb{C}(x) \cap W$.

Set $z = yx^{-1} \in \mathbb{C}(x)$ so that $y = zx$. Since x is central in $\mathbb{C}(x)$ we have $y^{\mathbb{C}(x)} = (z^{\mathbb{C}(x)})x$; so

$$k \geq |z^G| \geq |z^{\mathbb{C}(x)}| = k$$

and $z^G = z^{\mathbb{C}(x)}$. Hence $[G : \mathbb{C}(z)] = [\mathbb{C}(x) : \mathbb{C}(x) \cap \mathbb{C}(z)]$ and since $G \neq \mathbb{C}(x)$ we conclude that $\mathbb{C}(z) \nsubseteq \mathbb{C}(x)$. Choose $g \in \mathbb{C}(z) - \mathbb{C}(x)$.

Then $y^g \in y^G = y^{\mathbb{C}(x)}$ so $y^g \in \mathbb{C}(x)$ and

$$y^g = (zx)^g = zx^g = y(x^{-1}x^g).$$

Now $y^g, y \in \mathbb{C}(x)$; so $x^{-1}x^g \in \mathbb{C}(x)$. Moreover by assumption $x^{-1}x^g \in W$ and so $x^{-1}x^g \in W \cap \mathbb{C}(x)$. Thus y and y^g are in the same coset of $W \cap \mathbb{C}(x)$ and by the above $y = y^g$. This yields $x = x^g$, a contradiction since $g \notin \mathbb{C}(x)$ and the result follows.

The following theorem is due to B. H. Neumann and J. Wiegold (see [8]). The proof given here is new.

THEOREM 4.4 [8]. *Let G be a group and let k be a positive integer.*

(i) *If $|G'| = k$, then $G = \Delta_k(G)$.*

(ii) *If $G = \Delta_k(g)$, then*

$$|G'| \leqslant (k^4)^{k^4}.$$

Proof. (i) is obvious since $x^G \subseteq G'x$.

We prove (ii) by induction on k. If G is Abelian, the result is trivial. Thus we may assume that G is non-Abelian and in particular that $k > 1$. Choose $x \in G - \mathbb{Z}(G)$. Now $[G : \mathbb{C}(x)] \leqslant k$ so let $x_1, x_2, \ldots, x_j$ be a set of coset representatives of $\mathbb{C}(x)$ with $j \leqslant k$ and $x_1 = x$. In the notation of Lemma 4.2 set

$$W = G_{x_1} G_{x_2} \cdots G_{x_j}.$$

Then W is normal in G and $|W| \leqslant (k^4)^{k^3}$ by that lemma. Moreover $W \supseteq (x, G)$ since $x = x_1$. Thus by Lemma 4.3 if $H = \mathbb{C}(x)/W \cap \mathbb{C}(x)$, then $H = \Delta_{k-1}(H)$. By induction

$$|H'| \leqslant (k-1)^{4(k-1)^4} \leqslant k^{4k^3(k-1)}.$$

Let $^-$ denote the natural map $G \to G/W$ and observe that

$$\overline{\mathbb{C}(x)} = \mathbb{C}(x)\, W/W \simeq \mathbb{C}(x)/W \cap \mathbb{C}(x) = H.$$

Now $G = \bigcup \mathbb{C}(x)x_i$ implies that $\bar{G} = \bigcup \overline{\mathbb{C}(x)}\, \bar{x}_i$ and also each $\bar{x}_i$ is central in $\bar{G}$ since $W \supseteq G_{x_i}$. It then follows easily that every commutator in $\bar{G}$ is in fact a commutator of elements in $\overline{\mathbb{C}(x)}$ and hence

$$|\bar{G}'| = |\overline{\mathbb{C}(x)}'| = |H'|.$$

Thus

$$|G'| \leqslant |W| \cdot |\bar{G}'| \leqslant (k^4)^{k^3} \cdot (k^4)^{k^3(k-1)} = (k^4)^{k^4}$$

and the theorem is proved.

COROLLARY 4.5. *Let G be a group. Then all conjugacy classes of G are of finite bounded size if and only if G' is finite.*

5. BOUNDED REPRESENTATION DEGREE

We now consider groups G with the property that G' is finite and central and such that $K[G]$ satisfies a polynomial identity. We do this by studying suitable irreducible representations of the algebra. The following is well known [6, Lemma 4.2].

LEMMA 5.1. *Let $E = K_m$ be the ring of $m \times m$ matrices over K and suppose that E satisfies a polynomial identity of degree n over K. Then $2m \leqslant n$.*

LEMMA 5.2. *Let D be a division algebra over a field K and suppose that*

$$\dim_K D < \text{cardinality of } K.$$

Then D is algebraic over K. In particular, if K is algebraically closed, then $D = K$.

Proof. Let $x \in D - K$. Then for all $z \in K$ we have $x - z \neq 0$ and hence $x - z$ is invertible. Clearly all such terms $(x - z)^{-1}$ commute. Since $\dim_K D$ is less than the cardinality of the set $\{(x - z)^{-1} \mid z \in K\}$ there must be a nontrivial linear dependence. Say

$$\sum_1^n \frac{w_i}{x - z_i} = 0$$

with $w_i, z_i \in K$, $w_i \neq 0$ and the z_i distinct. Multiplying the above by $\Pi(x - z_i)$ yields a nontrivial polynomial satisfied by x. Thus D is algebraic over K. Since K is central in D the second statement is clear.

LEMMA 5.3. *Let G be a group such that G' is a cyclic central q-group for some prime q and let K be a field whose characteristic is not q. Suppose $K[G]$ satisfies a polynomial identity of degree n. Then*

$$[G : \mathbb{Z}(G)] \leqslant (n/2)^2.$$

Proof. Let F be a field extension of K which is algebraically closed and with

$$\text{cardinality of } G < \text{cardinality of } F.$$

By Lemma 3.2, $K[G]$ satisfies a multilinear polynomial f of degree n. If we view f as a polynomial over F, then clearly $F[G]$ satisfies f. Thus replacing K by F if necessary we may assume that K is algebraically closed and

$$\text{cardinality of } G < \text{cardinality of } K.$$

If G is Abelian the result is trivial; so assume that $G' \neq \langle 1 \rangle$ and let $w \in G'$ have order q. We observe now that $1 - w \notin JK[G]$ where the latter is the Jacobson radical of $K[G]$. This follows from a lemma of Amitsur [6, Lemma 16.9] which yields

$$K[G'] \cap JK[G] \subseteq JK[G']$$

and from the fact that $JK[G'] = 0$ by assumption on the prime q. Hence there exists an irreducible representation ρ of $K[G]$ with $\rho(1 - w) \neq 0$. Since G' is a cyclic q-group and $\rho(w) \neq \rho(1)$ it follows that ρ is faithful on G'.

Let V be the corresponding irreducible $K[G]$-module and let D be the commuting ring. If $\dim_D V > n/2$, then it follows from the Density theorem, that for some positive integer $m > n/2$, $\rho(K[G])$ has a subalgebra S with K_m as a homomorphic image. Thus K_m satisfies a polynomial identity of degree n, a contradiction by Lemma 5.1. Hence $\dim_D V = t \leqslant n/2$ and again by the Density theorem $\rho(K[G]) \cong D_t$. Now D is a division algebra over K and

$$\dim_K D \leqslant \dim_K K[G] = \text{cardinality of } G < \text{cardinality of } K.$$

Thus by Lemma 5.2, $D = K$ and $\rho(K[G]) \cong K_t$ for some integer $t \leqslant n/2$.

For each $x \in G$ let $\chi(x)$ denote the trace of the matrix $\rho(x) \in K_t$ where we have taken a fixed identification of $\rho(K[G])$ with K_t. Let $Z = \mathbb{Z}(G)$. If $x \in Z$ then $\rho(x)$ is central in $\rho(K[G]) = K_t$ and hence $\rho(x) = a_x I$ where $a_x \in K$ and I is the identity matrix. Thus $\chi(x) = a_x t$. Since ρ is faithful on G' we see that $a_x \neq 1$ for $x \in G'$, $x \neq 1$.

Now assume $x \notin Z$. Then there exists $y \in G$ which does not commute with x. Hence $y^{-1}xy = zx$ for some $z \in G'$, $z \neq 1$. Since similar matrices have the same trace we obtain

$$\begin{aligned} \chi(x) &= \chi(y^{-1}xy) = \operatorname{Tr} \rho(zx) \\ &= \operatorname{Tr} a_z \rho(x) = a_z \chi(x), \end{aligned}$$

and hence $\chi(x) = 0$ since $a_z \neq 1$.

We have shown that χ vanishes off of Z and that for each $x \in Z$, $\chi(x)$ is a multiple of t. If $t = 0$ in K this would yield $\chi(x) = 0$ for all $x \in G$. But then every matrix of $\rho(K[G]) = K_t$ would have trace zero, a contradiction. Therefore, $t = \chi(1) \neq 0$ in K.

Let $x_1, x_2, \ldots, x_r$ be a finite set of elements in distinct cosets of Z. We show that $\rho(x_1), \rho(x_2), \ldots, \rho(x_r)$ are linearly independent over K. Suppose that

$$b_1\rho(x_1) + b_2\rho(x_2) + \cdots + b_r\rho(x_r) = 0$$

for $b_i \in K$. Multiplying the above on the left by $\rho(x_i^{-1})$ we obtain

$$\sum_{j=1}^{r} b_j \rho(x_i^{-1}x_j) = 0.$$

Now for $j \neq i$, $x_i^{-1}x_j \in G - Z$ and thus $\rho(x_i^{-1}x_j)$ has trace $\chi(x_i^{-1}x_j) = 0$. Therefore, by taking traces of the above expression we obtain $b_i\chi(1) = 0$ and hence $b_i = 0$ since $\chi(1) \neq 0$. This shows that the $\rho(x_i)$'s are linearly independent and thus $r \leqslant t^2$. This clearly yields

$$[G : Z] \leqslant t^2 \leqslant (n/2)^2,$$

and the result follows.

THEOREM 5.4. *Let G be a group such that G' is central and $|G'| = k < \infty$. Let K be a field and suppose that $K[G]$ satisfies a polynomial identity of degree n. Then G has a characteristic subgroup A with $[G : A] \leqslant (n/2)^k$ such that*

(i) *A is Abelian if K has characteristic 0.*

(ii) *A is p-Abelian if K has characteristic $p > 0$.*

Proof. Let us assume first that $|G'| \neq 0$ in K. Also the result is trivial for $k = 1$; so we assume that $k > 1$. Since G' is finite and Abelian, it follows that G' is a direct product of cyclic groups of prime power order and clearly there are at most $k/2$ such factors. This implies that there exists normal subgroups $N_1, N_2, \ldots, N_t$ of G' with $t \leqslant k/2$, $\bigcap_1^t N_i = \langle 1 \rangle$ and with G'/N_i cyclic of prime power order. Now G' is central in G; so N_i is normal in G.

Let $G_i = G/N_i$. Since $K[G] \to K[G/N_i] = K[G_i]$ is an epimorphism it follows that $K[G_i]$ satisfies a polynomial identity of degree n. Moreover, $G_i' = G'/N_i$ is a cyclic central q-group for some prime q not equal to the characteristic of K. Thus by Lemma 5.3, $[G_i : \mathbb{Z}(G_i)] \leqslant (n/2)^2$. Let Z_i be the complete inverse image of $\mathbb{Z}(G_i)$ in G. Then $[G : Z_i] \leqslant (n/2)^2$ and $(G, Z_i) \subseteq N_i$. Thus if $A = \bigcap_1^t Z_i$, then

$$[G : A] \leqslant (n/2)^{2 \cdot k/2} = (n/2)^k$$

and $(G, A) \subseteq \bigcap N_i = \langle 1 \rangle$. Since $(G, A) = \langle 1 \rangle$ and since clearly $\mathbb{Z}(G) \subseteq Z_i$ we conclude that $A = \mathbb{Z}(G)$; so A is a characteristic Abelian subgroup of G.

This clearly yields (i). Now we consider the general case of (ii). Let K have characteristic $p > 0$ and let P be the normal Sylow p-subgroup of G'. Then P is normal in G and set $\bar{G} = G/P$. Clearly, $K[\bar{G}]$ satisfies a polynomial identity of degree n and $|\bar{G}'| = |G'/P|$ is not zero in K. Thus by the above $\bar{G}$ has a characteristic Abelian subgroup $\bar{A}$ with $[\bar{G} : \bar{A}] \leqslant (n/2)^k$. Let A be the complete inverse image of $\bar{A}$ in G. Then $[G : A] = [\bar{G} : \bar{A}]$ and $A' \subseteq P$ so A is p-Abelian. Finally, A is characteristic in G since P is and the result follows.

6. CONCLUSION

It is now a simple matter to put all the pieces together.

Proof of Theorem 1.1 (ii) *and Theorem* 1.3 (ii). Let $K[G]$ satisfy a polynomial identity of degree n. Set

$$a = a(n) = (n!)^2, \qquad b = b(n) = a^{4^{(a+1)!}}.$$

Then by Corollary 3.5 G has a characteristic subgroup G_0 with

$$[G : G_0] \leqslant (a + 1)!, \qquad G_0 = \Delta_b(G_0).$$

Set

$$c = c(n) = (b^4)^{b^4}, \qquad d = d(n) = (n/2)^c.$$

Then by Theorem 4.4, $|G_0'| \leqslant c$. Let $G_1 = \mathbb{C}_{G_0}(G_0')$. Then $G_1' \subseteq G_0'$; so G_1' is a finite central subgroup of G_1. Moreover

$$|G_1'| \leqslant c, \qquad [G_0 : G_1] \leqslant c!\,.$$

Since $K[G_1]$ also satisfies a polynomial identity of degree n we can apply Theorem 5.4 to obtain a subgroup A of G_1 with

$$[G_1 : A] \leqslant d, \qquad |A'| \leqslant c.$$

Moreover, if K has characteristic 0, then A is Abelian, and, if K has characteristic $p > 0$, then A is p-Abelian. Since

$$[G : A] \leqslant (a + 1)!\, c!\, d, \qquad |A'| \leqslant c,$$

the results follow.

It is interesting to observe that at each step of the above proof a characteristic subgroup is obtained and therefore A is in fact a characteristic subgroup of G.

We remark that the main results of this paper offer affirmative answers to problems (1)–(3) and (5) of [6]. However, as is evident, the bounds for $[G : A]$ and $|A'|$ we obtain here are astronomical. For this reason the module-theoretic techniques of [2] and [6] should not be abandoned. Rather, an attempt should be made to synthesize the two approaches in order to hopefully obtain more realistic bounds.

Finally, the methods of this paper with minor modifications also yield necessary and sufficient conditions for a twisted group ring to satisfy a polynomial identity. A discussion of these results and of the modifications needed will appear later.

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